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SECONDARY FLOWS IN AN UNSTABLE BOUNDARY LAYER

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Analysis of a large amount of experimental data on the structure of the transition region from laminar to turbulent flow on a planar plate [1, 2] makes it possible to draw the conclusion that, following the stages of linear development of the original instability and weak nonlinear development of perturbations, their three-dimensional growth takes place, and at some moment the evolution of wave motion is inevitably three-dimensional. Experiments determine the existence of long-wave eddy formation with the axis along the main flow direction, as a result of which there is a redistribution of mean flow momentum and the appearance of a secondary three-dimensional regime. The experimentally observed longitudinal vortices are periodic in the coordinate z, as is schematically illustrated in Fig. 1. Significant progress has been achieved in the study of the nonlinear transition stages, which cannot be said on analysis of three-dimensional effects as applied to the mean flow.

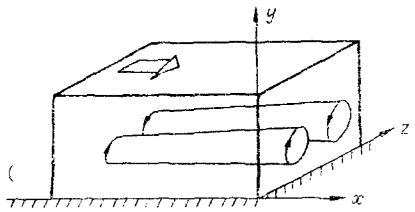

Fig. 1

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The first attempts of describing the mechanism of the three-dimensional interaction were undertaken [3, 4] for shear flows, modeling real flows of the boundary layer type. These longitudinal periodic vortices are now well known under the name "Benny–Lin vortices." The model suggested by them (a linear approximation) made it possible to obtain the mean secondary flow induced by perturbations. Similar assumptions were used in [5], where a boundary layer near the experimental one (the Falkner–Skan family) was considered as a reference. An important feature of these studies is the treatment of wave interaction of different dimensionality (planar and three-dimensional) with a reference flow, and the study of vortex shape changes for various amplitude relations of these waves. The distortion of the averaged reference flow was not treated. A more complete study of the flow in the field of three-dimensional finite intensity perturbations in an incompressible flow on a planar plate is carried out in the present study on the basis of the exact solution of the Reynolds equations for both the average and vortex flows.

Although the problem of competition of wave interaction of different dimensionality perturbations is also very interesting, it is commonly assumed that the presence of three-dimensional waves is dominant in the general pattern of appearance of vorticity. In the present study we consider their interaction with the reference planar boundary layer, selected in the form of the Falkner–Skan solution for a vanishing longitudinal pressure gradient. The method used here does not exclude, in principle, further complication of the problem in the given direction. As a reference model we selected symmetric wave crossing, a pair of intersecting sloping Tollmien–Schlichting waves, making it possible to express simply force moments in the general interaction pattern. We also investigate in the linear approximation the time evolution of the spectrum of the Orr–Sommerfeld equation, which may be useful for estimating pertubation phase velocities of the three-dimensional flow obtained.

1. Consider two perturbation waves in the form of sloping Tollmien–Schlichting waves of arbitrary initial intensity $\varkappa$. We assume for simplicity that the waves propagate with identical phase velocities $C = C_r + iC_i$ and wave numbers α, γ

$$\varkappa\{u', v', w', p'\}(x, y, z, t) = \varkappa\{u, v, w, p\}(y)\exp(\theta \pm i\gamma z),\quad \theta = i\alpha(x - Ct). \tag{1.1}$$

If γ is real, one may construct a linear combination of these waves with α and C vanishing at several points of the z axis, i.e., a solution of standing wave type. The basic equations of motion are invariant with respect to simultaneous changes of sign of z and w, therefore the following relations are valid for the perturbation amplitudes:

$$u_1(y) = u_2(y) = u(y),\quad v_1(y) = v_2(y) = v(y),\quad w_1(y) = -w_2(y) = w(y),\quad p_1(y) = p_2(y) = p(y),$$

while the resultant perturbation wave is represented in the form

$$u_1' + u_2' = u(y)\exp\theta(\exp i\gamma z + \exp(-i\gamma z)) = 2u(y)\exp\theta\cdot\cos\gamma z,$$
$$v_1' + v_2' = 2v(y)\exp\theta\cdot\cos\gamma z,\quad w_1' + w_2' = 2iw(y)\exp\theta\cdot\sin\gamma z,$$
$$p_1' + p_2' = 2p(y)\exp\theta\cdot\cos\gamma z.$$

If the initial flow is plane-parallel, $U = U(y)$, $V = W = 0$, the linearized Navier–Stokes equations for the perturbation amplitudes are [6]

$$v_{yy} - Bv = \mathrm{Re}\, p_y,$$
$$u_{yy} - Bu = \mathrm{Re}\, U_y v + i\alpha\,\mathrm{Re}\, p,\quad w_{yy} - Bw = i\gamma\,\mathrm{Re}\, p,$$
$$v_y + i\alpha u + i\gamma w = 0,\quad B = \alpha^2 + \gamma^2 + i\alpha\,\mathrm{Re}\,(U - C), \tag{1.2}$$
$$\mathrm{Re} = U_0\delta/\nu,\quad v_y = dv/dy.$$

The boundary conditions for them are sticking conditions at the wall

$$u = v = w = 0,\quad y = 0$$

and finite perturbation condition at infinity

$$u,\ v,\ w \to 0,\ y \to \infty.$$

In the present study we consider a self-similar boundary layer on a planar plate, described by the Falkner–Skan equation [7]:

$$\varphi_{yyy} = -\varphi\varphi_{yy} \tag{1.3}$$

with boundary conditions

$$\varphi = \varphi_y = 0\ (y = 0),\ \varphi_y \to 1\ (y \to \infty).$$

The boundary value problem reduces to the Cauchy problem

$$\varphi = \varphi_y = 0,\ \varphi_{yy} = 0.4696\ (y = 0),$$

and the dimensionless velocities are expressed in the form

$$U = \varphi_y,\ U_y = \varphi_{yy}.$$

The equation is solved in the interval $0 \leqslant y \leqslant \delta$, where the width of the boundary layer δ is selected by the condition $U(\delta) = 0.9999$ ($\delta = 4.85$). Using the Squire transformations, we reduce the three-dimensional problem to an equivalent two-dimensional one. Retaining the nonvarying phase velocity C and v = v, we introduce new variables u_0, p_0, χ, k, R as follows:

$$k^2 = \alpha^2 + \gamma^2,\ \ ku_0 = \alpha u + \gamma w,\ \ k\mathrm{R} = \alpha\mathrm{Re},\ \ \chi = \gamma u - \alpha w,\ \ p_0\mathrm{R} = p\mathrm{Re}.$$

We then obtain for the perturbation amplitudes the equivalent Orr–Sommerfeld system of equations

$$\begin{gathered} v_{yy} - Av = \mathrm{R}p_{0y},\ u_{0yy} - Au_0 = \mathrm{R}U_yv + ik\mathrm{R}p_0, \\ v_y + iku_0 = 0,\ A = k^2 + ik\mathrm{R}(U - C) \end{gathered} \tag{1.4}$$

and an inhomogeneous equation for χ

$$\chi_{yy} - A\chi = \gamma\ \mathrm{Re}\ U_yv \tag{1.5}$$

with boundary conditions

$$u_0 = v = \chi = 0\ (y = 0),\ u_0,\ v,\ \chi \to 0\ (y \to \infty). \tag{1.6}$$

The solution of system (1.4), (1.6) is the subject of study of the linear theory of hydrodynamic stability. A large number of reliable methods have been developed so far to obtain both eigenvalues k, R, C and eigenfunctions [8]. For a boundary layer of finite width the conditions of perturbation damping at infinity are replaced by continuity and finite perturbation conditions at the outer boundary layer, which acquire the form

$$\begin{gathered} \varepsilon_1 \equiv u_{0yyy} + (k + \beta)u_{0y} + k\beta u_0 = 0, \\ \varepsilon_2 \equiv v_{yy} + (k + \beta)v_y + k\beta v = 0,\quad \beta^2 = k^2 + ik\mathrm{R}(1 - C). \end{gathered} \tag{1.7}$$

The solution of (1.5) can also be obtained by one of the methods mentioned, and the following relation is valid for χ in the outer boundary within first approximation:

$$\chi_y + \beta\chi = 0.$$

2. Assuming that at all stages of evolution the perturbations satisfy the linearized Navier–Stokes equations, the quasistationary flow satisfies the Reynolds equations

$$UU_x + VU_y + WU_z + \frac{1}{\rho}P_x - \nu\Delta U = -(\langle u'^2\rangle_x + \langle u'v'\rangle_y + \langle u'w'\rangle_z),$$

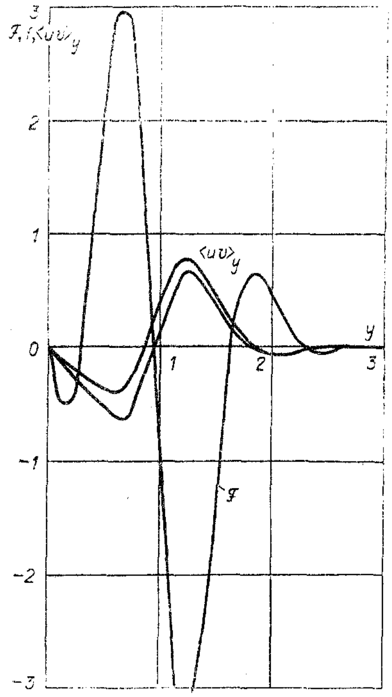

Fig. 2

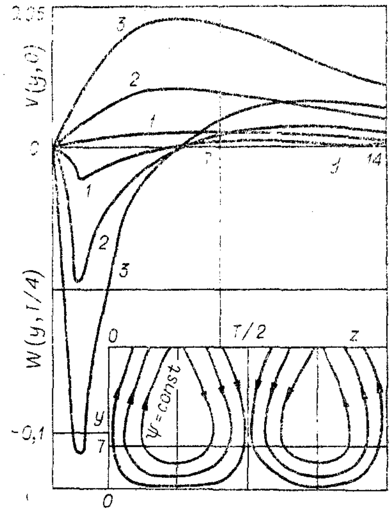

Fig. 3

$$UV_x + VV_y + WV_z + \frac{1}{\rho} P_y - \nu\Delta V = -(\langle u'v'\rangle_x + \langle v'^2\rangle_y + \langle v'w'\rangle_z),$$
$$UW_x + VW_y + WW_z + \frac{1}{\rho} P_z - \nu\Delta W = -(\langle u'w'\rangle_x + \langle v'w'\rangle_y + \langle w'^2\rangle_z),$$
$$U_x + V_y + W_z = 0,\ \Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2, \qquad (2.1)$$

where the right hand sides contain the Reynolds stresses, obtained by statistical averaging of the quantities u'^2, $v'w'$, etc. The problem of closing (2.1) was considered within the monoharmonic approximation [8]. For the selected system of waves (1.1) we obtain that the Reynolds stresses are independent of the coordinate x; for example,

$$\langle u'^2\rangle = 2|u|^2 \cos^2 \gamma z,\ \langle u'w'\rangle = 2(u_i w_r - u_r w_i) \sin 2\gamma z.$$

The resultant perturbation wave represents a crossing with nonvanishing momentum (y, z), therefore the force couple associated with it creates longitudinal vorticity $\omega = V_z - W_y$ in the boundary layer, and the flow acquires the form $U = U(y, z)$, $V = V(y, z)$, $W = W(y, z)$. For this flow the system (2.1) can be separated, and a solution can be found for ω independently of U.

In the variables ω, ψ ($V = -\psi_z$, $W = \psi_y$) the three last equations of (2.1) are written in the form

$$-\psi_z\omega_y + \psi_y\omega_z = (1/\mathrm{Re})(\omega_{yy} + \omega_{zz}) + \varkappa^2 F(y, z),$$
$$\omega = -(\psi_{yy} + \psi_{zz}), \qquad (2.2)$$

where

$$F(y,\ z) = -\mathcal{F}(y) \sin 2\gamma z; \mathcal{F}(y) = \langle vw\rangle_{yy} + 2\gamma(\langle vv\rangle_y + \langle ww\rangle_y + 2\gamma\langle vw\rangle).$$

Since the perturbations have a structure periodic in z, it can be expected that the secondary flow will also be periodic in z with the same period $T = 2\pi/2\gamma$. Consider the linear approximation of the full problem (2.2), called here a "viscous vortex." The linear problem describes well the form of vortices for low perturbation intensities, when the nonlinear convective terms are smaller than the viscous linear ones; it can also be considered as a fixed approximation of the total problem for arbitrary amplitudes. These restrictions are also reflected in [3-5]. Using formal Fourier series expansions, we have for the first term

$$\psi(y, z) = \Psi(y) \sin 2\gamma z. \tag{2.3}$$

In that case $\omega = -(\Psi_{yy} - 4\gamma^2\Psi) \sin 2\Psi z$, and from (2.2) we obtain for $\Psi(y)$ a fourth-order inhomogeneous equation

$$\Psi_{yyyy} - 2(2\gamma)^2\Psi_{yy} + (2\gamma)^2\Psi = -\varkappa^2\mathcal{F}(y).$$

The boundary conditions follow by physical considerations

$$V = W = 0 \quad \text{for} \quad y = 0, \text{ which gives } \Psi = \Psi_y = 0,$$
$$V, W \to 0 \quad \text{for} \quad y \to \infty, \text{which gives } \Psi, \Psi_y \to 0.$$

The latter condition refers to the outer boundary layer, where one can adopt a decreasing shape for ψ in the form $\psi = (D_1 + D_2 y) \exp(-2\gamma y)$, leading to relations similar to (1.7):

$$\Psi_{yy} + 4\gamma(\Psi_y + \gamma\Psi) = 0, \ \Psi_{yyy} + 4\gamma(\Psi_{yy} + \gamma\Psi_y) = 0.$$

The secondary flow velocity components are determined from (2.3)

$$V(y, z) = -2\gamma\Psi(y) \cos 2\gamma z, \ W(y, z) = \Psi_y(y) \sin 2\gamma z.$$

The solution of the full problem (2.2) in the region $G(0 \leqslant y \leqslant Y,\ 0 \leqslant z \leqslant T)$ was found by using an implicit finite-difference method of second order accuracy [9]. Continuity conditions were required to be satisfied at the left and right boundaries ($z = 0$, $z = T$):

$$\psi(y, 0) = \psi(y, T), \ \omega(y, 0) = \omega(y, T),$$
$$\psi_z(y, 0) = \psi_z(y, T), \ \omega_z(y, 0) = \omega_z(y, T).$$

The boundary condition at the wall ($y = 0$) $V = 0$ makes it possible to determine $\psi(0, z) =$ const; in particular, one can take $\psi(0, z) = 0$. Satisfaction of the second condition $W = 0$ was verified in the convergence process of the solution. For the vorticity ω we have from the second equation of (2.2) $\omega(0, z) = -\psi_{yy}(0, z)$. Since the vorticity of linear perturbation waves decays quickly with increasing y, it must be expected that the vorticity of secondary flow induced by them can also be taken equal to zero for sufficiently large y. The upper boundary of the calculated region was widely varied, $\delta \leqslant Y \leqslant 3\delta$; it seemed that ω was practically equal to zero for $y \sim 1.5\delta$ already.

The stream function also decays at infinity, but considerably more slowly. Therefore, to restrict the region of solution we use asymptotic relations. For the boundary conditions stated we use the periodicity conditions of ψ in $z(\psi, (y, z) = \psi(y) \sin 2\gamma z)$, and from the second of Eqs. (2.2) and the condition $\omega(Y, z) = 0$ we obtain for the amplitude function $\psi(y)$ $\psi_{yy} - 4\gamma^2\psi = 0$. If it is now assumed that the asymptotic $\psi(y)$ satisfying this equation is of the form $\psi(y) \sim D \exp(-2\gamma y)$, where D is an arbitrary constant, a relation of the form $\psi_y + 2\gamma\psi = 0$ can be adopted as a boundary condition as $y = Y$. More accurate results can give an asymptotic representation of $\psi(y, z)$ for large y in the form of a harmonic series

$$\psi(y, z) = \sum_{i=1}^{n} D_i \exp(-2\gamma iy) \sin(2\gamma iz).$$

The solutions obtained for boundary conditions in a series form with $n > 3$ and the relation $\psi_y + 2\gamma\psi = 0$ are very close. The distribution (2.3) was assigned as initial approximation. To obtain the longitudinal velocity components of the main flow $U(y, z)$ one must solve the first equation of system (2.1) ($P_x = 0$ for a plate):

$$VU_y + WU_z = (1/\mathrm{Re})(U_{yy} + U_{zz}) - \varkappa^2 H,$$

where H is the Reynolds stress

$$H = f + \langle uv\rangle_y, \ f = (\langle uw\rangle_y + 2\gamma\langle uw\rangle) \cos 2\gamma z.$$

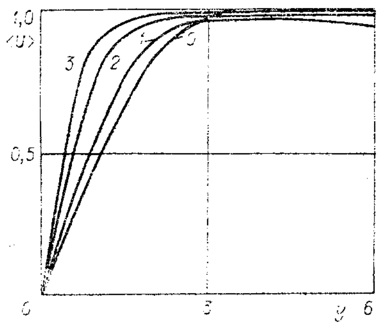

Fig. 4

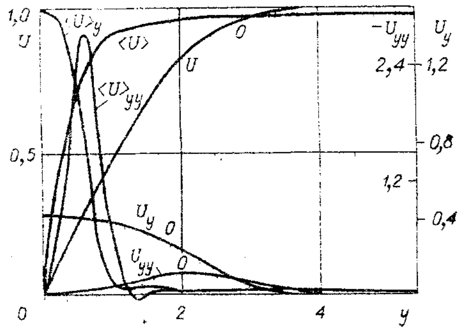

Fig. 5

To obtain hence velocity curves U(y, z) = U(y) corresponding to the distribution (1.3) in a laminar boundary layer without perturbations (or for infinity small perturbations with $\varkappa^2 \to 0$) and without secondary shapes induced by them (V = W = 0), a supplementary force $F_a = -(1/\mathrm{Re})\varphi_{yyy}$ was introduced into the right-hand side of this equation, having the form of a Falkner–Skan force (1.3). The Reynolds equation, thus, acquires the form

$$VU_y + WU_z = (1/\mathrm{Re})(U_{yy} + U_{zz}) - \varkappa^2 H + F_a. \tag{2.4}$$

The solution range is the rectangle G_1 ($0 \leqslant y \leqslant Y_1$, $0 \leqslant z \leqslant T$), and for $Y = Y_1$ the regions G and G_1 coincided. The quantity Y_1 was also varied, $\delta \leqslant Y_1 \leqslant 3\delta$. The following boundary conditions held for U(y, z):

$$U(y, 0) = U(y, T),\ U_z(y, 0) = U_z(y, T),\ U(0, z) = 0,\ U_y(Y_1, z) = 0,$$

and a possible variant of the last condition is the asymptotic expansion $U(Y_1, z) = 1 + \sum_{i=1}^{n} D_i \exp(-2\gamma Y_1 i) \cos(2\gamma z i)$. The distribution (1.3) was chosen as initial approximation. To solve (2.4) we used an implicit scheme of second order accuracy (the Pisman–Ruckford scheme) [10].

A study of the evolution of the time spectrum of the Orr–Sommerfeld equation (1.4) for the secondary averaged flow $\langle U(y)\rangle = \frac{1}{T}\int_0^T U(y, z)\,dz$ was performed by methods of linear stability theory. A zero of the function ε_2 of (1.7) was sought on the outer boundary Y_1. This function is analytic in its argument $C = C_r + iC_i$, therefore the use of the Cauchy–Riemann condition simplified the calculation considerably. The averaged flow <U> and its derivatives were approximated by cubic spline functions.

3. The results presented below were obtained for critical Reynolds numbers in the linear theory ($k_1 = 0.301$, $R_1 = 519.2$, $C = 0.3959 + i0$ ($k = 0.24745$, $R = 426.7$) for two-dimensional perturbations for different values of the transverse wave numbers $\gamma = 0.05$; 0.1; 0.15 (0.2k, 0.4k, 0.6k respectively). The transition to three-dimensional perturbation parameters is realized by the Squire equations, and $k_1 = k\delta_1$, $R_1 = R\delta_1$.

Figure 2 shows the Reynolds stress amplitudes $\mathscr{F}$, f, $\langle uv\rangle_y$ for $\gamma = 0.1$. The integral curve $\mathscr{F}(y)$ is negative, and, consequently, the rotation generated by it must occur clockwise. For the same values Fig. 3 shows the amplitude of secondary "viscous vortices" (linear approximation), Y was selected $\sim 3\delta$ to obtain closed forms of vortices, and the stream lines Ψ = const are illustrated also. The curves 1-3 correspond to the perturbation intensities $\varkappa = 0.01$, 0.02, and 0.03. The secondary vortex is symmetric in z, its center is found at y = 4, z = T/4 for all $\varkappa$, and two vortices with right and left rotation are located on the period $0 \leqslant z \leqslant T$.

The average secondary flow <U(y)> is represented in Fig. 4 for $0.01 \leqslant \varkappa \leqslant 0.03$ (curves 1-3) with step $\Delta\varkappa = 0.01$. The subscript 0 denotes the laminar profile (1.3). It is seen from Fig. 4 that the secondary longitudinal vortices, induced in the boundary layer by three-dimensional perturbations, lead to a significant flow rearrangement; the profile becomes more filled in the boundary layer region. The emergence on the velocity of the unperturbed

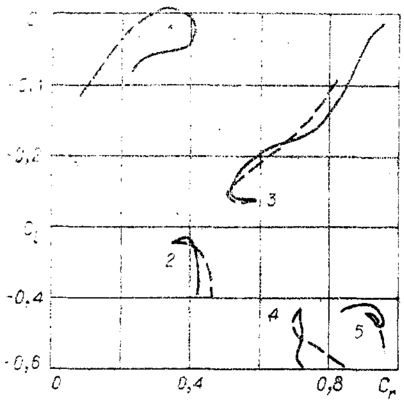

Fig. 6

flow occurs for y ∿ 4–5δ, where δ is the original width (1.3).

Figure 5 shows the change in the resultant characteristics of averaged flow <U>, $<U>_y$, $<U>_{yy}$ for ϰ = 0.05, compared with laminar forms denoted by the subscript 0. It seems that all characteristic quantities change in the boundary layer y < 3, as this is also a peculiar feature of turbulent flow. Similar results were obtained for other values of γ (0.05 and 0.15), while both the vortex intensity and profile occupation increase with γ for the same ϰ. Interesting results were obtained in studying the time evolution of the spectrum of the Orr–Sommerfeld equation within the linear approximation. It was found in advance that for the critical Reynolds number in a laminar boundary layer there exist six spectral modes, the first of which is neutrally stable (C_i = 0). The data of [11] served for verification in this attempt. In the flow evolution process this pattern is rearrangement, as shown in Fig. 6 for γ = 0.05 (dashed lines) and γ = 0.1 for ϰ = 0.05; mode 1 for γ = 0.05 occurs in terms of weakly unstable setting of C_i > 0, and then, as are also modes 2–5, they become "strongly" stable (large negative C_i). The behavior of the third mode is interesting. For γ ⩾ 0.1 the value of C_i for the established profile is very close to zero (γ = 0.1, C = 0.96–0.007i). It can be concluded that the possible instability of secondary flow can be related to perturbations whose phase velocity is of the order of the leading flow velocity, as is characteristic of a vortex forming boundary layers observed experimentally (U ∿ 0.8 U_0). For γ = 0.05 this is not very distinct. The sixth mode is not traced for the secondary flow. One of the calculation variants for ϰ = 0.02 is shown in Figs. 7, 8. Figure 7 shows the quantities V and W in different cross sections z, and for comparison we plot the maximum amplitudes of the "viscous" approximation. The vortices obtained are displaced and deformed in y and z in comparison with the "viscous" ones.

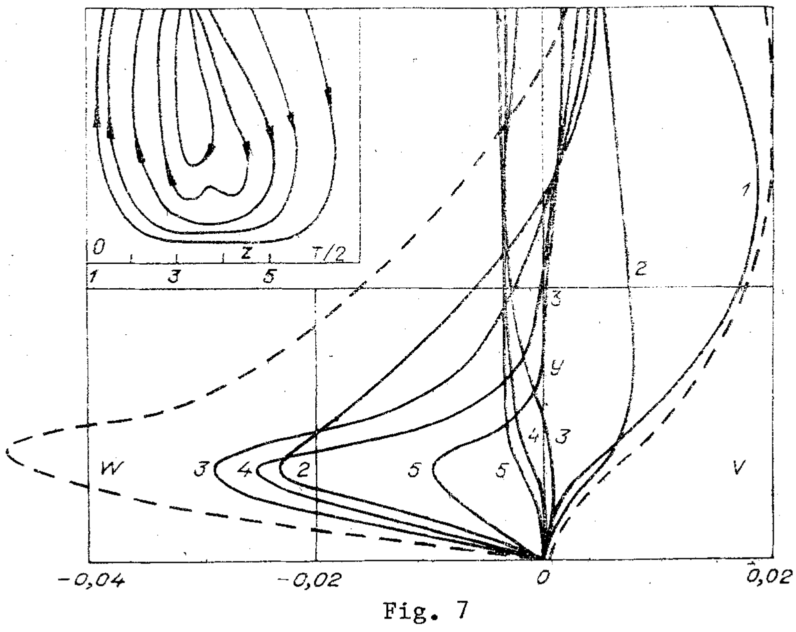

Fig. 7

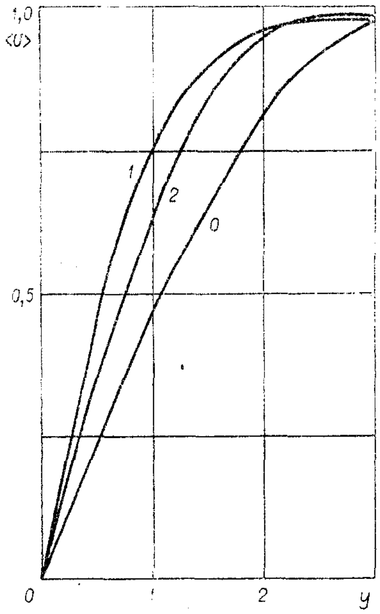

Fig. 8

The results make it possible to clarify the effect of the convective terms. If for $\varkappa \leqslant 0.01$ the values of V and W of both approximations are very close (for $\varkappa = 0.005$ the deviation is $W_{max} \sim 3\%$, and the amplitudes of V practically coincide), for intense perturbations $\varkappa > 0.01$ the velocity amplitudes of the full vortex are significantly smaller, i.e., energy expenditures on convective mixing promote vortex quenching. Similar smoothing is also seen on the average velocity profile (Fig. 8), and leads to smaller profile occupation in the boundary layer region (here 0 is the laminar profile, 1 is the viscous approximation, and 2 is the full problem.

An attempt has been made to calculate the degree of turbulence of the flow obtained ε_T. Similarly to power-law turbulence we take $\varepsilon_T = ((1/3)(\langle u'^2\rangle + \langle(V + v')^2\rangle + \langle(W + W')\rangle))^{1/2}$. It seems that though we obtained $\varepsilon_T = \varepsilon_T\ (y,\ z)$, this value is very close to the initial value of the perturbation intensity $\varkappa$. Thus, it can be assumed that within the model adopted flow turbulization is determined by the perturbation intensity.

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